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Mathematical modeling of fruit trees' growth under scarce watering

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Abstract. As a consequence of the Global Warming, the decrease in water resources is a problem that affects the agricultural systems worldwide. In particular, the Global Warming has produced a sustained decrease in precipitation in Mediterranean-type climatic zones and the more frequent presence of droughts. A dry-and hot environment could negatively affect the physiology of the fruit tree, compromising its growth and development rates, especially, the yield of a fruit tree results from the interactions between genotype, climate, soil, crop management, and irrigation. This work theoretically studies the growth dynamics of fruit trees with a limited water resource by a mathematical model. A system of non-linear ordinary differential equations was used as a method to describe the growing phenomenon. A qualitative analysis of the system was carried out, and local and global stability results were obtained. The analysis showed that the limitation of water resources compromises the growth dynamics of the fruit tree, causing a total loss of the fruit tree.

1. Introduction

Rising global temperatures due to global warming are significantly affecting agricultural systems of Mediterranean-type climate zones worldwide. This increase in average temperatures, added to heatwaves, has been coupled with a decrease in rainfall. Global warming is a big concern for fruit producers because, according to Food and Agriculture Organization of the United Nations (FAO), agricultural systems worldwide consume nearly 70% of the available freshwater, where the production of fruits is done mainly under irrigation. Thus, a decrease in the water resources, due to global warming, especially in semi-arid zones, generates a loss of productivity, affecting the physiology of the fruit tree and compromising its growth rate, fruit yield, and quality.

The changes in the surface energy balance due to Global warming have increased the high temperatures. In the trees growing season,the higher temperatures affect their water fluxes, increasing their water consumption or evapotranspiration. The higher evapotrans-piration, combined with dry soils, produces water stress to trees. This physiological response of the trees significantly reducing the tree's photosynthetic rates, negatively affecting the fruit growth [1–3]. It is important to highlight that fruit tree yields result from interactions among the genotype, climate, soil, crop management, and irrigation, which is the most critical factor. They have been extensively studied in field experiments for different conditions. However, the results presented are limited only to each empirical research, not applying to all possible scenarios.

A model is a representation of a real set with a certain degree of precision and as complete as possible [4], in which variables and parameters that describe the relationships to be studied are incorporated. To biological systems such as crops, a culture model can be described as a quantitative scheme for predicting growth, development and performance of this, given a set of genetic characteristics and environmental relevant variables [5, 6].

Different researches on crop modeling, have been done mainly in herbaceous crops as presented in [6–9]. Different models of dynamic type for the growth of trees are presented in [10]. However, as far as the authors are aware, there is no evidence of a mathematical model of fruit trees grow that considers the interaction between energy and water. This paper proposes a mathematical model to represent the growth dynamics of fruit trees under a water deficit scenario.

2. Mathematical modeling

The growth dynamics of a crop are the result of interactions involving the genotype, energy, soil, water, and crop management among other factors [6, 11]. therefore, the following assumptions are made.

(i) For the construction of the model proposed in this work that describes the growth dynamics of fruit trees under a water deficit scenario, an adult fruit trees (older than 5 years) is assumed with ideal soil conditions and optimal pest control management and the use of fertilizers under the recommendations made for that purpose.

(ii) We assume that the accumulated energy $E = E(t)$ in the system is constant, so the variation of this with respect to time $E'(t) = 0$ implies that $E(t) = q$.

The variables and parameters that intervene in the growth dynamics of fruit trees according to our model are summarized in Table 1.

Table 1. Parameters used in the model and their meaning.

Parameter	Meaning
q	Accumulated energy constant
r	Fruit trees intrinsic growth rate
N	Fruit carrying capacity
τ	Evapotranspiration rate
γ	Photosynthetic contribution rate
ω	Mortality rate of fruit trees

We denote $W = W(t)$ the water amount in soil in time t and $C = C(t)$ the fruit biomass concentration in time t, both state variables. The variation of the amount of water in the system with respect to the time is denoted by $W'(t)$. It is being affected by a loss of water determined by evapotranspiration at a rate τ. At the same time, another amount of water is being utilized for the growth of fruit trees at a rate r. Also, the variation of biomass growth in the system with respect to time denoted by $C'(t)$ is being affected by a corresponding input water-growth biomass interaction at a rate r. There is also a rate γ corresponding to the energy absorbed by the fruit trees for its vegetative growth, and there is an output given by the term ωC which corresponds to the loss of fruit trees growth due to natural causes at a rate ω.

The diagram that represents the growth dynamics of the fruit tree is presented in Figure 1; where E = energy; W = water concentration; C = fruit biomass concentration; r = fruit trees intrinsic growth rate; N = fruit carrying capacity; τ = evapotranspiration rate; γ = photosynthetic contribution rate; ω = mortality rate of fruit trees; p = rainfall rate; I = irrigation.

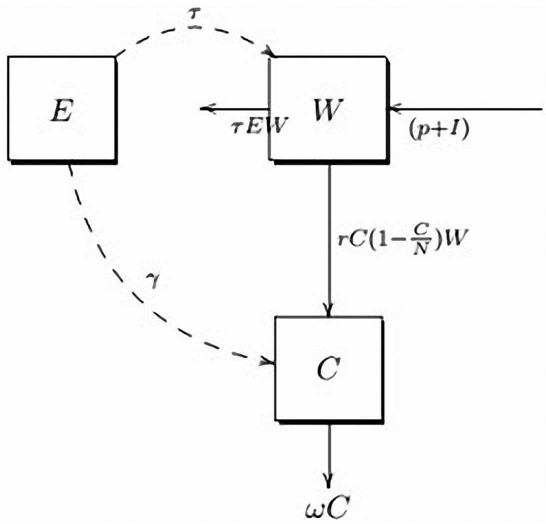

Figure 1. Dynamics flow chart.

Based on the aforementioned, this model proposal represents the growth dynamics of fruit trees under a water deficit scenario, which can be described by the system of non-linear differential equations, Equation (1), as follows.

$$\begin{cases} W'(t) &=& -\tau q W(t) - rC(t)(1-\frac{C(t)}{N})W(t), \\ C'(t) &=& rC(t)(1-\frac{C(t)}{N})W(t) + \frac{\gamma q C(t) W(t)}{(C+1)(W+1)} - \omega C(t), \end{cases} \tag{1}$$

where $W = W(t)$, $C = C(t)$ are nonnegative, we restrict our attention to the nonnegative quadrant $\Omega = \{(W,C) \in \mathbb{R}^2 : W \geq 0,\, C \geq 0\}$, and the parameters are all positive in the space $\rho = \{(q, r, N, \tau, \gamma, \omega) \in \mathbb{R}^6_+\}$.

3. Results

In this section, we consider the mathematical analysis of Equation (1). The technical results show the behavior of the dynamics studied.

Lemma 3.1. The set $\Omega = \{(W,C) \in \mathbb{R}^2 : W \geq 0,\, C \geq 0\}$ is positively invariant under the flow of Equation (1).

Proof. Since the Equation (1) is of Kolmogorov-type and is defined in Ω, the W-axis and C-axis are invariant sets.

Lemma 3.2. For initial conditions in Ω, the solutions of Equation (1) are uniformly bounded.

Proof. We define a function, Equation (2).

$$X(t) = W(t) + C(t). \tag{2}$$

The time derivative of Equation (2) along the solutions of Equation (1) is Equation (3).

$$\frac{dX}{dt} = -\tau q W - rC(1-\frac{C)}{N})W + rC(1-\frac{C}{N})W + \frac{\gamma q C W}{(C+1)(W+1)} - \omega C. \tag{3}$$

In Equation (3) the function $g_1 = \frac{\gamma q C W}{(C+1)(W+1)}$ is bounded, this is $|g_1(A,C)| \leq K\|(A,C)\|$, with $K > 0$. Then, we have Equation (4).

$$\frac{dX}{dt} \leq K - (\tau q W + \omega C). \tag{4}$$

Equation (4) is a differential inequality. Let $\delta = max\{\tau q, \omega\}$ and substituting in the Equation (4) we have Equation (5).

$$\frac{dX}{dt} \leq K - \delta X \tag{5}$$

Applying the theory of diferential inequalities [12] and comparison theorem [13] to solve the equation, we obtain Equation (6).

$$X(W,C) \leq \frac{K}{\delta}(1 - e^{-\delta t}) + X(W(0), C(0))e^{-\delta t}. \tag{6}$$

In Equation (6) for $t \to \infty$, we have $0 < W < \frac{K}{\delta}$. Hence all the solutions of Equation (1) is bounded in Ω.

Lemma 3.3. $P_e(W,C) = (0,0)$ is the unique equilibrium point in the set Ω.

Proof. The equilibrium points are the positive solutions of the system of nonlinear equations given in Equation (7).

$$\begin{aligned} -\tau q W(t) - rC(t)(1 - \tfrac{C(t)}{N})W(t) &= 0 \\ rC(t)(1 - \tfrac{C(t)}{N})W(t) + \tfrac{\gamma q C(t)W(t)}{(C+1)(A+1)} - \omega C(t) &= 0 \end{aligned} \tag{7}$$

From the first equation, we have Equation (8).

$$W = 0 \qquad or \qquad \left[-\tau q - rC(1 - \frac{C}{N})\right] = 0 \tag{8}$$

Equation (8) we obtain. If $W = 0$, then $C = 0$. Therefore $(W,C) = 0$ is an equilibrium point. If $\left[-\tau q - rC(1 - \frac{C}{N})\right] = 0$, This is Equation (9).

$$C^2 - CN - \frac{\tau q N}{r} = 0 \tag{9}$$

Then the roots of Equation (9) are real and are given by $C_1 = \frac{1}{2}(N + \sqrt{(-N)^2 + 4m})$ and $C_2 = \frac{1}{2}(N - \sqrt{(-N)^2 + 4m})$, with $m = \frac{\tau q N}{r}$. Substituting C_1 in the second term of the Equation (7), we have Equation (10).

$$W^2 + aW - b = 0. \tag{10}$$

With $a = 1 + \frac{N(\gamma q - \omega(C_1+1))}{r(N-C_1)(C_1+1)}$ and $b = \frac{\omega N}{r(N-C_1)}$. The roots of Equation (10) are real if $a^2 \geq 4b$ and are given by $W_1 = \frac{1}{2}(-a + \sqrt{a^2 + 4b})$ and $W_2 = \frac{1}{2}(-a - \sqrt{a^2 + 4b})$. This implies that: If $a > 0$, then W_1 and W_2 are negative, since $b < 0$. If $a = 0$, then the roots of Equation (10) are complex. If $a < 0$, then W_1 and W_2 are negative. Therefore, the unique non-negative equilibrium point of biological interest is $P_e = (0,0)$.

Proposition 3.1. The equilibrium point $P_e(W,C) = (0,0)$ is locally asymptotically stable.

Proof. We use the Lyapunov indirect method [14]. The Jacobian matrix of the Equation (1) is given in Equation (11).

$$J(\bar{W},\bar{C}) = \begin{pmatrix} -\tau q - r\bar{C}(1-\frac{\bar{C}}{N}) & -r\bar{W}(1-\frac{2\bar{C}}{N}) \\ r\bar{C}(1-\frac{\bar{C}}{N}) + \frac{\gamma q \bar{C}}{(\bar{C}+1)(\bar{W}+1)^2} & r\bar{W}(1-\frac{2\bar{C}}{N}) + \frac{\gamma q \bar{W}}{(\bar{W}+1)(\bar{C}+1)^2} - \omega \end{pmatrix}. \tag{11}$$

The matrix of linearization around the equilibrium P_e is given in Equation (12). This is, the Jacobian matrix in P_e.

$$J(P_e) = \begin{pmatrix} -\tau q & 0 \\ 0 & -\omega \end{pmatrix}. \tag{12}$$

The eigenvalues of $J(P_e)$ are $\lambda_1 = -q\tau$ and $\lambda_2 = -\omega$, both negative. Then the equilibrium point $P_e(W,C) = (0,0)$ is locally asymptotically stable.

Proposition 3.2. Equation (1) has no closed orbits in Ω.

Proof. We use the Bendixson's-Dulac's negative criterion [15]. Equation (1) is rewritten in the Equation (13).

$$\begin{cases} W'(t) &= F(W,C) \\ C'(t) &= G(W,C) \end{cases} \tag{13}$$

Define a Dulac function $B(W,C) = W^{-1}C^{-1}$, from Equation (13) we have the Equation (14).

$$\begin{aligned} D &= \frac{\partial(BF)}{\partial W} + \frac{\partial(BG)}{\partial C} \\ &= -\left(\frac{r}{N} + \frac{\gamma q}{(W+1)(C+1)^2}\right). \end{aligned} \tag{14}$$

The Equation (14) is strictly negative in the interior of the first quadrant. Thus, there cannot be a closed orbit in Ω.

Theorem 3.1. The equilibrium point $P_e(W,C) = (0,0)$ is globally asymptotically stable.

Proof. We use the Lyapunov Theorem [13]. Define a Lyapunov function $L(W,C) = (W+C)^2$ for Equation (1) that satisfies:

(i) $L(0,0) = 0$, for $(W,C) = (0,0)$.
(ii) $L(W,C) = (W+C)^2 > 0,\ \forall (W,C) \in \{\Omega - (0,0)\}$.
(iii) $L'(W,C) \leq 0$, this is Equation (15).

$$\begin{aligned} L(W,C) &= (W+C)^2, \\ L'(W,C) &= 2(W+C)W' + 2(W+C)C' \\ &= 2(W+C)(W'+C') \\ &= -2(W+C)\Phi. \end{aligned} \tag{15}$$

With $\Phi(W,C) = (\tau q W + \omega C - \frac{\gamma q W C}{(W+1)(C+1)})$, since $\Phi > 0$ for all $(W,C) \in \Omega$. Then in the Equation (15) $L'(W,C) < 0\ \forall (W,C) \in \{\Omega - (0,0)\}$, since $L(W,C)$ is a strict Lyapunov function. Therefore, $P_e(W,C) = (0,0)$ is globally asymptotically stable.

Figure 2 shows the dynamic behavior of the water amount (W) and the resulting biomass of the fruit tree (FB), considering a scenario of the variation in the accumulated energy of the system. In this respect, [W1] and [W2] depicts that the trend of water concentration sustained decreased in time, showing a higher depletion in the case of fruit concentration (fruit growth). Furthermore, [FB1] and [FB2] show that a higher water reduction produces higher fruit growth. Segmented lines show that when considering an increase in the accumulated energy, the concentration of water is depleted in a short period of time [W2], for its part the biomass of the fruit tree [FB2] increases rapidly, its growth compromising it is at the point in which there is water shortage.

These simulations could be considered realistic because the growth rate of fruit is affected by the transpiration rates of the fruit trees. Also, the trends of fruit growth present a logistic behavior, which is common in agricultural systems. According to [10], the logistic models allow describing that the growth is highly dependant on the quantity of growth machinery, which is proportional to the dry mass production. In this case, the growth is irreversible and directly dependent on water availability.

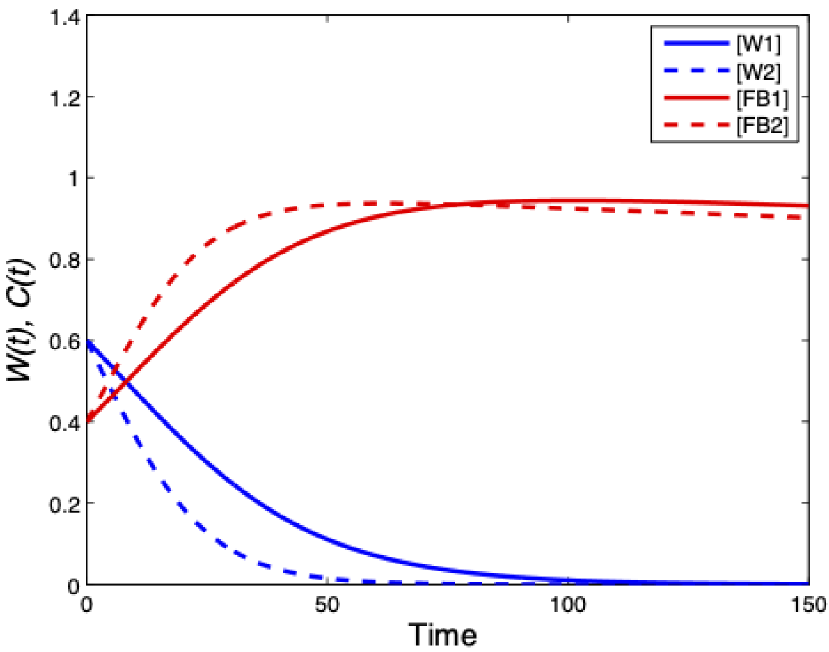

Figure 2. Simulations of the dynamic behavior of the water amount (W) and the resulting biomass of the fruit tree (C).

In this regard, the numerical simulation presented in Figure 2 is a good representation of the dynamic behavior of fruit trees growing in a water deficit scenario and is consistent with the studies presented in [2, 3]. In Figure 2 continuous lines corresponds to the dynamic of the fruit grow, and water depletion with $W = 0.6$, $C = 0.4$ and parameter values $r = 0.05$, $q = 0.07$, $N = 3000$, $\tau = 0.004$, $\gamma = 0.001$, $\omega = 0.0005$; while the segmented lines corresponds to the dynamic of the fruit grow, and water depletion considering an increase in accumulated energy with $W = 0.6, C = 0.4$ and parameter values $r = 0.09$, $q = 0.9$, $N = 3000$, $\tau = 0.004$, $\gamma = 0.001$, $\omega = 0.0005$. Moreover, the numbers 1 and 2 denote the first, and second simulations of (W) and (FB), respectively.

4. Conclusions

In this work, we developed a qualitative model, based on a bidimensional non-linear differential equations system, to describe the growth dynamics of a fruit tree, under a water deficit scenario.

The results of this work showed the vital importance of water resources for the growth of fruit trees. This was reflected in the technical results about asymptotical behavior, where was possible to prove that the origin is a global attractor for all feasible initial condition. The simulations of the sustained decrease in water availability generated losses in the productivity of the fruit trees, comprising their growth and development rates.

The performance of the proposed model was considered. Dynamical aspects related to system solutions qualitatively reflected the effects of water depletion on the fruit trees' growth.

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References

[1] Shan Q, Wang Z, Ling H, Zhang G, Yan J, Han F 2021 Unreasonable human disturbance shifts the positive effect of climate change on tree-ring growth of Malus sieversii in the origin area of world cultivated apples *Journal of Cleaner Production* **287** 125008

[2] Lévesque M, Siegwolf R, Saurer M, Eilmann B, Rigling A 2014 Increased water-use efficiency does not lead to enhanced tree growth under xeric and mesic conditions *New Phytologist* **203(1)** 94

[3] Ogaya R, Barbeta A, Başnou C, Peñuelas J 2015 Satellite data as indicators of tree biomass growth and forest dieback in a Mediterranean holm oak forest *Annals of Forest Science* **72(1)** 135

[4] Hernández N, Soto F, Caballero A 2009 Modelos de simulación de cultivos: características y usos *Cultivos Tropicales* **30(1)** 73

[5] Monteith J L 1996 The quest for balance in crop modeling *Agronomy Journal* **88(5)** 695

[6] Steduto P, Hsiao T C, Raes D, Fereres E 2009 AquaCrop—The FAO crop model to simulate yield response to water: I. Concepts and underlying principles *Agronomy Journal* **101(3)** 426

[7] Pezzini R V, Cargnelutti Filho A, de Bem C M, de Souza J M, Chaves G G, Neu I M M, Procedi A 2019 Gompertz and logistic models for morphological traits of sudangrass cultivars during sowing seasons *Semina: Ciências Agrárias* **40(6Supl3)** 3399

[8] Roberts M J, Braun N O, Sinclair T R, Lobell D B, Schlenker W 2017 Comparing and combining process-based crop models and statistical models with some implications for climate change*Environmental Research Letters* **12(9)** 095010

[9] Chimonyo V G P, Modi A T, Mabhaudhi T 2015 Perspective on crop modelling in the management of intercropping systems*Archives of Agronomy and Soil Science* **61(11)** 1511

[10] Thornley J H, Johnson I R 1990 *Plant and Crop Modelling* (Clarendon: Oxford)

[11] Keating B A, Thorburn P J 2018 Modelling crops and cropping systems—evolving purpose, practice and prospects *European Journal of Agronomy* **100** 163

[12] Birkhoff G, Rota G C 1982 *Ordinary Differential Equations* (Boston: Ginn)

[13] David G, Schaeffer C, John W 2018 *Ordinary Differential Equations: Basics and Beyond* (New York: Springer)

[14] Chicone C 2006 *Ordinary Differential Equations with Applications* vol 34 (New York: Springer Science & Business Media)

[15] Kot M 2001 *Elements of Mathematical Ecology* (New York: Cambridge University Press)